THEORY OF THE HYDROMAGNETIC GENERATOR

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It is shown that in a homogeneous medium, a magnetic field may generate helical motion in a cylinder with constant angular and axial velocities. The generation problem is solved exactly, and analytic expressions for the magnetic field are found. At high velocities the increment of field growth is maximal when the ratio of the velocities is of the order of unity. The maximum increment and frequency are of the order of the velocity to the two-thirds power. The field distribution has the form of a surface wave. The field decay decrement for departure from the cylinder surface is proportional to the square root of its increment.

Concrete examples of self-excitation of a field are of interest in the theory of the hydromagnetic generator. Simple field forms (two-dimensional or axisymmetric) cannot be generated, and finding such examples is difficult. In [1] an exact solution of the generator problem is given. Below we offer an exact solution in which, in contrast to [1], the magnetic field varies with time.

The motion in the example considered is axisymmetric. Such motion cannot generate an axisymmetric field [2], but, as was shown in [3, 4], it can excite a nonsymmetric field.

The generation equations for a medium with magnetic viscosity equal to unity are

$$\operatorname{div} \mathbf{H} = 0, \quad \partial \mathbf{H} / \partial t = \operatorname{rot} [\mathbf{V} \times \mathbf{H}] + \Delta \mathbf{H} \tag{1}$$

where the velocity **V** is given.

In cylindrical coordinates r, φ, z, let the velocity components be $V_r = 0$, $V_\varphi = r\omega(r)$, $V_z = v(r)$. Then from Eq. (1), for a magnetic field proportional to $\exp(im\varphi + ikz + pt)$, we obtain

$$DH_r + H_r / r + imH_\varphi / r + ikH_z = 0 \tag{2}$$

$$LH_r - 2imH_\varphi / r^2 = 0 \tag{3}$$

$$LH_\varphi + 2imH_r / r^2 + rH_r D\omega = 0 \tag{4}$$

$$L_0 H_z + H_r Dv = 0$$

$$(D = d / dr, \ L + 1 / r^2 = L_0 = (1 / r)DrD - m^2 / r^2 - q^2 \tag{5}$$

$$q^2 = s^2 + i\mu, \ s^2 = p + k^2, \ \mu = m\omega + kv)$$

The solutions of Eqs. (2)-(5) must be finite, continuous, and tend to zero as $r \to \infty$. The field is generable if there exist eigenvalues p with a positive increment $\gamma = \operatorname{Re} p$. Generation is impossible if one of the parameters m, k, v is equal to zero [2, 5].

We will now consider the case where ω, v are constant for r < 1 and equal to zero for r > 1. From Eqs. (3), (4) it is evident that the quantities $H_\pm = H_r \pm iH_\varphi$ satisfy the equations $(L \mp 2m/r^2) H_\pm = 0$. The finite single-valued solutions of these equations are

$$H_\pm = \begin{cases} A_\pm I_\pm(qr) / I_\pm(q), & r < 1 \\ B_\pm K_\pm(sr) / K_\pm(s), & r > 1 \end{cases} \quad (|\arg q, s| \leqslant {}^1/_2 \pi) \tag{6}$$

Here and below I, K, $I_\pm$, $K_\pm$ are modified Bessel functions with indices m, m ± 1.

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The constants $A_{\pm}$ and $B_{\pm}$ are determined from the condition that

$$H_{\pm},\ DH_{\pm} \pm {}^1/_2 i\omega (H_+ + H_-) \tag{7}$$

be continuous at r = 1.

The continuity of the field and the quantities $DH_\varphi + \omega H_r$, $DH_z + vH_r$ follows from Eqs. (7), (2), (3). The latter must be continuous; this follows from the continuity of the tangential electric field, and directly from Eqs. (4), (5) after integration over r from 1 − 0 to + 0.

From Eqs. (6), (7) we obtain the dispersion relation

$$ {}^1/_2 i\omega (R_+ - R_-) = R_+R_- \qquad (R_{\pm} = qI_{\pm}' / I_{\pm} - sK_{\pm}'/ K_{\pm}) \tag{8}$$

The number p is an eigenvalue if it satisfies Eq. (8) and the condition $|\arg s| \le {}^1/_2\pi$, necessary for finiteness of the field (6).

For large q, s, in Eq. (8) the asymptotic relationships [6]

$$\begin{gathered} \sqrt{2z/\pi}\, K(z) = e^{-z} (1 + a_1/z + a_2/z^2 + \ldots) \equiv f(z), \\ \sqrt{2z\pi}\, I(z) = f(-z) \\ (n!2^n a_n = \Gamma(m + n + {}^1/_2) / \Gamma(m - n + {}^1/_2), \quad |\arg z| < {}^1/_2\pi) \end{gathered} \tag{9}$$

may be used, from which follows

$$R_{\pm} = q + s + ({}^1/_2 m^2 \pm m + {}^3/_8)(q^{-1} + s^{-1} + q^{-2} - s^{-2}) + O(q^{-3} + s^{-3}) \tag{10}$$

With accuracy to the largest terms in Eqs. (8), (10), we have

$$im\omega (1/q + 1/s) = (q + s)^2 \qquad (|\arg q,\ s| < {}^1/_2\pi) \tag{11}$$

It is sufficient to study Eqs. (8), (11) at $\alpha = ({}^1/_2 m\omega)^{1/3} \ge 0$, since they become complex conjugates for replacement of ω, μ, p by $-\omega$, $-\mu$, $\bar{p}$.

At $\mu = 0$ (when a point of the surface $r = c \le 1$ moves along the spiral with constant field component values), from Eq. (11) we obtain

$$s = q = \alpha \exp({}^1/_6 i\pi) \qquad (\alpha \to +\infty) \tag{12}$$

The field is generable if $\alpha^2 > 2k^2$.

For small $\delta = \mu/(2\alpha^2)$ the quantity $\rho = (p + k^2)\alpha^{-2} + i\delta$ is determined from Eqs. (8), (10), (12) by the perturbation method:

$$\begin{gathered} \rho = \rho_0 - [{}^2/_3 \rho_0 \delta / \alpha + {}^5/_{12} \delta^2 / \rho_0 + O(\alpha^{-2})]\,[1 + O(\delta^2) + O(\alpha^{-2})], \\ \rho_0 = \exp({}^1/_3 i\pi) \end{gathered}$$

From this it follows that the increment γ is maximum at $\mu = -{}^8/_5\alpha + O(\alpha^{-1})$.

For large δ (when $1/\alpha \ll |\delta| \ll \alpha/m^2$), the value ρ is determined from Eq. (11) or the equivalent equation $4 + 4\rho(\rho^2 + \delta^2) = \delta^2(\rho^2 + \delta^2)^2$. According to Eq. (11), with increase in δ^2 the increment $\gamma(\delta) = \alpha^2 \mathrm{Re}\rho(\delta) - k^2$ decreases. This is evident from the monotonicity of the functions

$$\begin{gathered} \delta^2 = 4a \frac{\sin({}^1/_4\pi - \varphi)}{\sin 4\varphi}, \quad \mathrm{Re}\, p = \frac{\mathrm{tg}\, 2\varphi}{a} \cos({}^1/_4\pi - \varphi) \\ a^3 = \frac{\sin({}^1/_4\pi + \varphi)}{\sin 4\varphi} \qquad \left(0 \le \varphi \le \frac{\pi}{12}\right) \end{gathered} \tag{13}$$

To derive Eq. (13), we multiply Eq. (11) by $(q-s)/(q+s)$ and write it in the form

$$T_- - T_+ = \delta \qquad (T_{\pm}^{-2} = \rho \pm i\delta)$$

From this it follows that

$$T_- + T_+ = 2i\,(T_-T_+)^2, \quad 2\rho = T_+^{-2} + T_-^{-2}.$$

Taking $T_\pm = T \mp \tfrac{1}{2}\delta$, we obtain

$$\sqrt{T/i} = T^2 - \tfrac{1}{4}\delta^2, \quad \rho = i\,(T + \tfrac{1}{4}\delta^2 / T)$$

Substituting $\sqrt{T} = \alpha \exp(-i\varphi)\ (\alpha > 0)$ and separating real and imaginary parts, Eq. (13) may be obtained.

We have considered above only the eigenvalue characterizing the field generated. A peculiarity of Eqs. (2)-(5), (7) is that the number of their eigenvalues is finite and nonconstant. With decrease in velocities ω, v eigenvalues disappear, since the roots p of Eq. (8) depart from the curve $|\arg s| \leq \tfrac{1}{2}\pi$ to neighboring ones through the section $p \leq -k^2$. At low velocities there are no eigenvalues.

To determine the disappearing eigenvalues and the corresponding frequencies in the case q = s it is convenient, using the equation [6]

$$I'K - IK' = 1/z, \quad I' \pm mI/z = I_\mp, \quad K' \pm mK/z = -K_\mp \tag{14}$$

to transform Eq. (8) to the form

$$1/R = IK, \quad 1/\omega = \tfrac{1}{2}i\,(1/R_- - 1/R_+) = -im\,(IK)'/s = -\tfrac{1}{2}\pi m\,(J^2 + iJY)'/x \tag{15}$$

where J, Y are Bessel functions of the first and second type with argument x = is.

An eigenvalue disappears if the corresponding root x moves from the upper semiplane into the lower across the positive semiaxis. For positive x it follows from Eq. (15) that

$$X = (JY)' - 2J'Y + 2/(\pi x) = 0, \qquad m\omega = x^2Y/J \tag{16}$$

The positive nulls x_n of the function X satisfy the inequalities

$$0 < x_1 < j_1' < y_1 < x_2 < y_1' < j_1 < x_3 < j_2' < y_2 < x_4 < y_2' < \ldots \tag{17}$$

since the sign of X is different at the boundaries of each interval for x_n. The latter may be verified by expansion of X for $x \to 0$ and from the inequalities (17) for the nulls j, j', y, y' of the functions J, J', Y, Y'.

From Eqs. (16), (17) and the asymptotic [6]

$$\sqrt{\tfrac{1}{2}\pi x}\,(J + iY) \sim \exp i\,(x - \tfrac{1}{2}\pi m - \tfrac{1}{4}\pi), \qquad j_n \approx \pi\,(n + \tfrac{1}{2}m - \tfrac{1}{4})$$

it develops that

$$x_n \approx \tfrac{1}{2}\pi\,(n + m - 1), \qquad m\omega_n \approx (-1)^n x_n^2 \qquad (n \gg m)$$

For m = 1 the first roots ≈ 0.6, 2.9, 4.6, and the frequencies ≈ −1.5, 6.6, −22.5. For m = 2, 3, 4 the first roots ≈ 1.8, 2.8, 3.8 and the frequencies ≈ −7.7, −18, −32.

For frequencies close to ω_n from Eq. (15) we can find the correction to x_n and verify that the eigenvalues disappear with decrease in $|\omega|$.

From Eqs. (16), (17) it follows that $\omega_n(-1)^n > 0$. In accordance with the remarks made about Eq. (11) above, for disappearing eigenvalues p in the upper semiplane $\omega_n(-1)^n < 0$. Therefore, with increase in ω from zero, the first eigenvalue appears in the upper semiplane, the second in the lower, etc. It may be assumed that Eq. (12) corresponds to the first eigenvalue.

In Eq. (1) with initial conditions, for Laplace transform of the magnetic field **H**, Eqs. (3)-(5) are obtained, to the left side of which have been added the corresponding components of the initial field **h**. Instead of Eq. (6) we have

$$H_{\pm} = A_{\pm} I_{\pm}(qr) / I_{\pm}(q) + K(qr) \int_0^r h_{\pm}(\xi) I_{\pm}(q\xi) \xi \, d\xi + + I_{\pm}(qr) \int_r^1 h_{\pm}(\xi) K_{\pm}(q\xi) \xi \, d\xi \quad (h_{\pm} = h_r \pm i h_{\varphi}, \quad 0 < r < 1) \tag{18}$$

The expression for the field at $r > 1$ is obtained from Eq. (18) by the replacement $A \to B$, $q \to s$, $0 \to 1$, $1 \to \infty$, and in the first term $I \to K$.

After determination of $A_{\pm}$, $B_{\pm}$ from Eq. (7), each component of Eq. (18) is presented in the form $H = H_*(p, r)/W(p)$, where $W = 0$ when Eq. (8) is satisfied. From this and the rotation integral we obtain

$$H(t, r) = \sum e^{pt} H_* / W' + \int e^{pt} H_* / (2\pi i W) \, dp$$

where the sum is taken over the eigenvalues, and the integral is over the curve bounding the section $|\arg s, q| = \frac{1}{2}\pi$. For large t the sum of the terms describing the generated field will be the greatest. If in the semi-plane $\operatorname{Re} p \geq -k^2$ there are no eigenvalues, then the integral term will be greatest, equal to $O[t^{-m} \exp(-k^2 t)]$ for $t \to \infty$. From this it is evident that generation is impossible if there are no eigenvalues.

The component H_z is found from Eqs. (2), (18). Using Eq. (14), we can find

$$H_z = \frac{i}{k} I(qr) \left[\frac{q}{2} \left(\frac{A_+}{I_+(q)} + \frac{A_-}{I_-(q)} \right) - cK(q) \right] + K(qr) \int_0^r h_z I(q\xi) \xi \, d\xi + I(qr) \int_r^1 h_z K(q\xi) \xi \, d\xi \; (0 < r < 1)$$

The expression for $r > 1$ is obtained by the replacement $q \to s$, $0 \to 1$, $1 \to \infty$, $A \to -B$, $c = h_r(1) \to -c$ and in the first term $I \rightleftharpoons K$.

The solution of the generator problem considered here may be generalized to the case where the velocities ω, v and the conductivity are arbitrary piecewise-constant functions of the radius. An exact dispersion relation can be written for the case differing from the above by a discontinuity in conductivity at $r = r_0 \neq 1$ (the cases of the boundaries of a conductor with a semiconductor and a vacuum are the limiting ones).

According to Eqs. (6), (9), at large s the magnetic field decays exponentially with departure from the cylinder $r = 1$; therefore inhomogeneity of conductivity (which is dependent not only on r) changes the eigenvalue of the homogeneous problem by an exponentially small value, if the area of inhomogeneous conductivity is removed from the cylinder by a minimum distance $\gg 1/\operatorname{Re} s$. In the problem with a discontinuity in conductivity, the change in the eigenvalue is proportional to $\exp(-2s|r_0 - 1|)$. In the example considered, inhomogeneity in conductivity is not significant for generation.

With increase in one of the dimensions of motion, difficulty in field generation is to be expected. In the limiting case of plane motion (when the velocity is dependent only on the Cartesian coordinate x and $V_x = 0$), generation is impossible.

It follows from considerations of continuity that generation remains possible for replacement of the moving cylinder by a long torus and smoothing of the velocity discontinuities. As in the examples of [3, 4], this confirms the possibility of generation of an axisymmetric-motion field under astrophysical conditions.

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